More on Adiabatic Invariants

→ for parameter $\lambda(t)$ s/t

$$\left.\dot{\lambda}(t)/\lambda < \omega\right] \rightarrow$$ multiple time scales separated.

$$\frac{d}{dt}\bar{I} = 0 \qquad \bar{I} = \oint_{E,\lambda \text{ fixed}} p\,dq$$

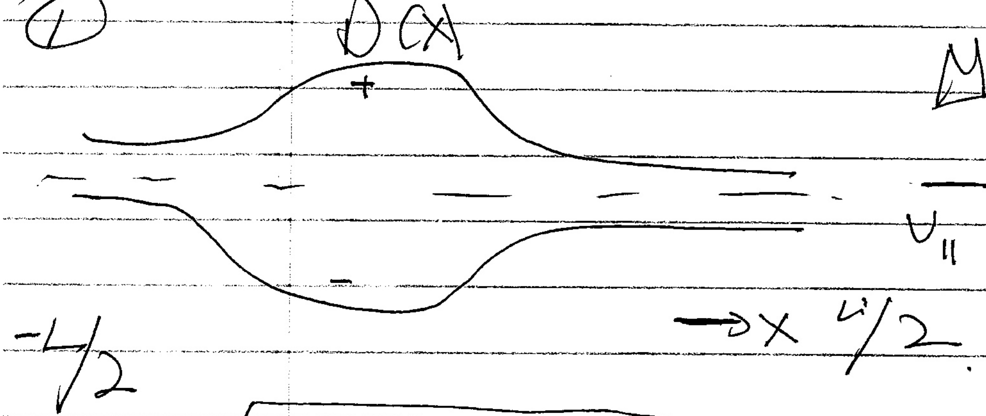
$\bar{I}$ → adiabatic invariant

→ $\boxed{\text{adiabatic invariance} \Leftrightarrow \text{phase symmetry, along } \oint. \text{ (i.e. can start anywhere in integration).}}$

Applications of Adiabatic Invariants

Consider 2 related non-trivial (adiabatic invariant-related) systems:

(1)



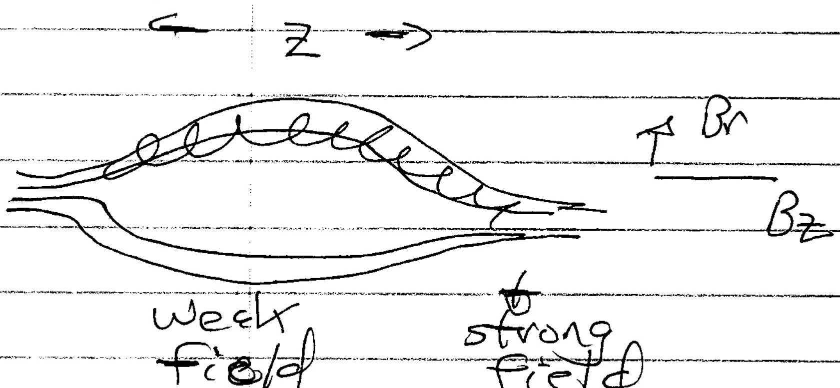
Mechanical Mirror

(i.e. curved hard surface)

c.f.: video

n.b. $\boxed{D/L \ll 1}$.

(2) Magnetic Mirror $\Rightarrow$ basis for mechanical mirror



$$\nabla \cdot \underline{B} = 0$$

$$\frac{\partial}{\partial z} B_z + \nabla_r B_r = 0$$

i.e. $\frac{\partial}{\partial z} B_z \neq 0 \Rightarrow \partial_r B_r \neq 0$

for "long, thin" mirror – anisotropy! $\Rightarrow$ { long thin $\Rightarrow$ slow axial variation

$$B_r \cong -\frac{r}{2}\frac{\partial B_z}{\partial z}\Big|_{r_0}$$

from: $B_r = -\frac{1}{r}\int_0^r dr'\, r' \frac{\partial B_z}{\partial z}$

Consider time scales:

→ $\tilde{\tau}_{b_\perp} \sim (v_\perp/2D)^{-1}$ ⇒ ⊥ bounce time

→ $\tau_{b_\parallel} \sim L/v_\parallel$ ⇒ parallel bounce time

i.e. ⊥ 

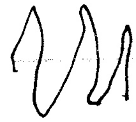
– 

so if consider

$\tau_{b_\perp} < t$ ⇒ - many bounces.
- sufficient time to sense curvature of D
- can define adiabatic invariant

$\int p_\perp dq_\perp$

$$2\pi I = \oint m v_\perp \, dy \to \oint p_\perp \, dq_\perp$$



$$= \int_{-D}^{D} dy\, \underbrace{m v_\perp}_{\text{forward}} + \int_{D}^{-D} \underbrace{(-m v_\perp)}_{\text{back}} dy$$

$$= 4 m D v_\perp$$

$$\boxed{I = \frac{2}{\pi} D m v_\perp}$$

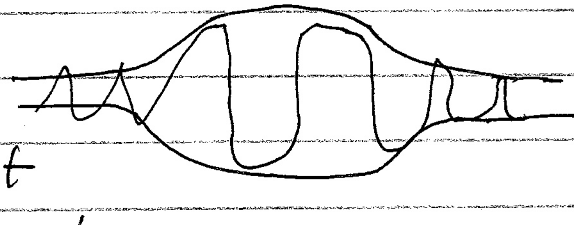
adiabatic invariant on times $t > \tilde{\tau}_{b_\perp}$

i.e. $D v_\perp \sim \text{const}$

$v_\perp \begin{cases} \text{large in throat} \\ \text{smaller in center} \end{cases}$

given initial $D(x_0) v_\perp(x_0)$, can determine $v_\perp(x)$ for all x.

Motion?



{particle can reflect from throat

energy conserved!

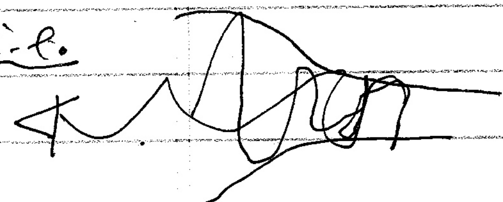
$$E = \tfrac{1}{2} m (v_\perp^2 + v_\parallel^2)$$

$$= \tfrac{1}{2} m \left(v_\parallel^2 + \frac{\pi^2}{4D^2} \frac{I^2}{m^2} \right)$$

$$\Rightarrow v_\parallel^2 = \frac{2E}{m} - \frac{\pi^2}{4D(x)^2} \frac{I^2}{m^2}$$

so if I s/t $\dfrac{\pi^2 I^2}{4D(x)^2 m^2} > \dfrac{2E}{m}$ → particle reflected in mirror throat

i.e.



$$I = \frac{2}{\pi} D(x_0)\, m v_{\perp 0}$$

frequently written as:

$$I = \frac{2}{\pi} D(0)\, m v_\perp(0)$$

$x_0 \Leftrightarrow$ center.

$$\frac{\pi^2 I^2}{4 D(x)^2 M^2} > \frac{2E}{m}$$

i.e.

$$\Rightarrow \left(\frac{D(x_0)}{D(x)}\right)^2 v_\perp^2(x_0) > \frac{2E}{m}.$$

i.e.

for $x < L$ $\Rightarrow$ particle will bounce!

As $E = \frac{1}{2} m \left(v_{\perp 0}^2 + v_{\parallel 0}^2 \right)$;

$$\Rightarrow \boxed{\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} < \left(\frac{D(0)}{D(L)}\right)^2 - 1}$$

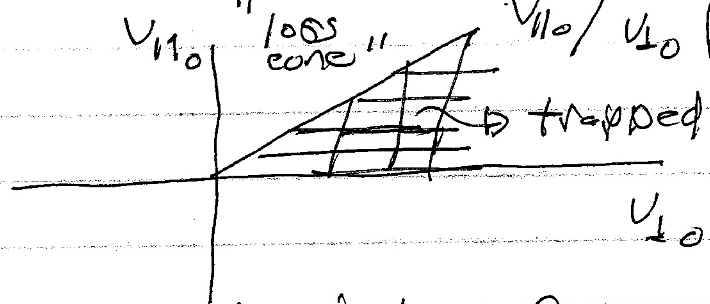
"mirror ratio"

i.e. optimal ratio

$$R_m \equiv \frac{D(x_0)^2}{D(x)^2} \rightarrow \frac{D(0)^2}{D(L)^2}$$

i.e. trapping condition

$v_{\parallel 0} / v_{\perp 0} \; (R_m - 1)^{1/2}$ (for $x_0 \sim 0$, $x \sim L$)



$$R_m \equiv \frac{D(0)^2}{D(L)^2}$$

basic description of mirror confinement

Now, can determine reflection point

simply by:

$$v_{\parallel}^2 = \frac{2E}{m} - \frac{\pi^2}{4D(x_R)^2}\frac{I^2}{m^2} = 0$$

defines

$x_R \leq L/2$

then: can envision longer times:

$$t > \tau_{b_\parallel} \gg \tau_{b_\perp}$$

$$\tau_{b_\parallel} = \oint \frac{dx}{|v_\parallel|}$$

parallel bounce time, for trapped particles

so

Can have "2^{nd}" adiabatic invariant on time scale $\tau_{b_\parallel} > \tau_{b_\perp}$

$$J_\parallel = \oint dx\, p_\parallel$$

"bounce invariant"

2^{nd}.

$J_\perp \Rightarrow$ first adiabatic inv.

$\Rightarrow \perp$ bounce.

i.e.

$$\tau_\perp < \tau < \tau_\parallel < \tau'$$

$\tau_\perp$: $\perp$ bounce time

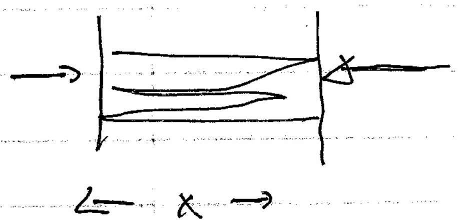
τ: time of $I = J_\perp$ invariant

$\tau_\parallel$: parallel bounce time

τ': time of $J_\parallel$ invariant

N.B.: Can expect 1 adiabatic invariant per closed cyclic orbit (n.b. cyclic orbit in action-angle sense).

For application of $J_\parallel$: [Adiabatic compression]



if push slowly:

$$J_\parallel = \oint p_\parallel \, dx = \text{const}$$

$$J_\parallel = \int_{-L}^{L} p_\parallel \, dx + \int_{L}^{-L} -p_\parallel \, dx$$

$$= p_\parallel (2L) - - p_\parallel (2L)$$

$$\delta J_\parallel = 0 \Rightarrow \delta (p_\parallel L) = 0$$

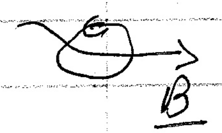
$$\Rightarrow \delta p_\parallel = - \delta L$$

(2) Magnetic Mirror

→ scheme is the same, with magnetic field variation as agent of confinement

→ now, for particle in magnetic field

$$\underline{p} \rightarrow \underline{p} - \frac{e}{c}\underline{A}$$

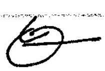


consider cyclotron orbit in plane ⊥ to field

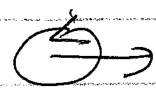
$$\oint_{\perp \text{ plane}} p\, dq = \oint_{\text{cycl}} p_\perp \cdot dq_\perp$$ ⇒ integrated along Larmor orbit.

$$= \int_c p_\perp \cdot dq_\perp - \frac{e}{c}\int_c A_\perp \cdot dq_\perp$$

$$= \int_c m v_\perp \cdot d\underline{\ell}_\perp - \frac{e}{c}\int_c A_\perp \cdot dq_\perp$$



Larmor disk



$$= m v_\perp (\rho_L 2\pi) - \frac{e}{c}\pi \rho_L^2 B$$

↓ $2\pi r$ with r = radius of Larmor disk

↳ flux thru Larmor disk.

so

$$\oint_\perp p\,dq = m v_\perp \frac{v_\perp}{\frac{eB}{mc}} 2\pi - \frac{e}{c}\pi B \frac{v_\perp^2}{\frac{e^2B^2}{m^2c^2}}$$

$$= \frac{m v_\perp^2}{2B}\left(4\pi \frac{Mc}{|e|}\right) - \frac{m v_\perp^2}{2B}\left(2\pi \frac{Mc}{|e|}\right)$$

$$= \frac{m v_\perp^2}{2B}\underbrace{\left(\frac{4\pi Mc}{|e|}\right)}_{\text{irrelevant const.}}$$

so

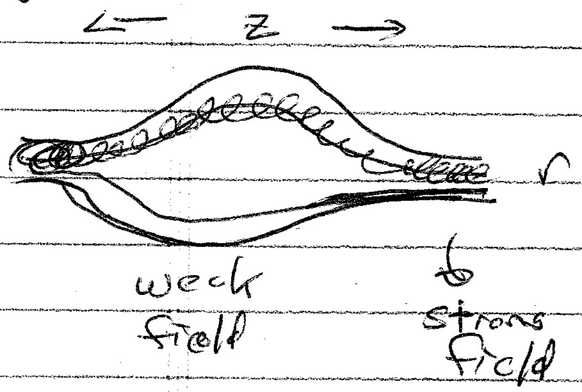
$$\oint_\perp p\,dq = \underbrace{\frac{m v_\perp^2}{2B}}_{\text{magnetic moment}}$$

Physically: — $\boxed{\text{Magnetic moment corresponds to action computed for 1 cyclotron orbit}}$

— adiabatic invariant on $t > T_{cycl}$, else approx. of closure of cyclotron orbit is meaningless.

Another approach

3.) Magnetic Mirror — basis for mechanical mirror



$$\underline{\nabla} \cdot \underline{B} = 0$$

$$\frac{\partial}{\partial z} B_z + \nabla_r B_r = 0$$

$\neq 0$

Now, consider rate of change of $\perp$ Energy

$$\frac{d}{dt}\left(\frac{m v_\perp^2}{2}\right) = q\,\underline{E}_\perp \cdot \underline{v}_\perp$$

avg over 1 cyclotron orbit $\Rightarrow$

$$\delta\left(\frac{m v^2}{2}\right) = \int_{\Omega^{-1}} dt\, q\, \underline{E}_\perp \cdot \underline{v}_\perp$$

$\underline{v}\,dt = \ell$

change in energy in 1 cyclotron orbit

$$= \int_{C_{\text{gyro circle}}} d\underline{\ell} \cdot \underline{E}_\perp q = q \int \underline{E} \cdot d\underline{\ell}$$

$\rho \to$ gyro-radius

$$= \int d\underline{a}\; q \cdot \underline{\nabla} \times \underline{E}$$

via Faraday

$$= -\int_{\text{gyrodisk}} d\underline{a} \cdot \left(\frac{q}{c}\frac{\partial \underline{B}}{\partial t}\right)$$

$$\cong -\pi \rho^2 \;\frac{q}{c}\frac{\partial B}{\partial t}$$

$$\rho^2 = V_\perp^2/\Omega^2$$

$\Rightarrow$

$$\delta\left(\frac{mV_\perp^2}{2}\right) \cong -\pi \frac{q}{c} \frac{V_\perp^2}{\frac{q^2 B^2}{m^2 c^2}} \frac{\partial B}{\partial t}$$

$$= -\frac{mV_\perp^2}{\Omega} \frac{\pi}{B} \frac{\partial B}{\partial t}$$

but $\underset{\substack{\downarrow \\ \text{change in} \\ \text{1 cyclotron} \\ \text{period}}}{\delta B} = \underset{\substack{\downarrow \\ \tau_c}}{\frac{2\pi}{\Omega}} \frac{\partial B}{\partial t}$

$$\delta\left(\frac{mV_\perp^2}{2}\right) = -\frac{mV_\perp^2}{2} \frac{1}{B} \quad \delta B$$

$$\Rightarrow \boxed{\delta\left(\frac{mV_\perp^2}{2B}\right) = 0}$$

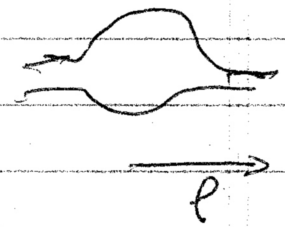
$\Rightarrow$ adiabatic time variation in $B \Rightarrow$ heating

so $\boxed{\mu = mV_\perp^2/2B}$ $\rightarrow$ magnetic moment adiabatic invariant on $t \gg \Omega^{-1}$

Now, for mirroring:

$$\cancel{\tfrac{1}{2}m}(V_{\parallel}^2 + V_{\perp}^2) = \cancel{\tfrac{1}{2}m}(V_{\parallel 0}^2 + V_{\perp 0}^2)$$

$$\frac{\cancel{m} V_{\perp}^2(0)}{\cancel{2}B(0)} = \frac{\cancel{m} V_{\perp}^2(\ell)}{\cancel{2}B(\ell)}$$



$$V_{\parallel}^2(0) + V_{\perp}^2(0) = V_{\parallel}^2 + \frac{B(\ell)}{B(0)} V_{\perp}^2(0)$$

$$V_{\perp}^2(0)\left(1 - \frac{B(\ell)}{B(0)}\right) = V_{\parallel}^2(\ell) - V_{\parallel}^2(0)$$

for confinement: $V_{\parallel}^2(\ell) = 0 \Rightarrow$

so

$$\boxed{\frac{V_{\parallel}^2(0)}{V_{\perp}^2(0)} < \frac{B(\ell)}{B(0)} - 1}$$

mirror ratio

obvious analogy to:

$$\frac{V_{\parallel 0}^2}{V_{\perp 0}^2} < \frac{D(x_0)^2}{D(x_R)^2} - 1$$

$B(\ell) \leftrightarrow 1/D(x_R)^2 \rightarrow$ strong $B \rightarrow$ frequent gyration frequent bouncing

$B(0) \leftrightarrow 1/D(x_0)^2 \rightarrow$ weak $B \rightarrow$ less frequent bouncing gyration.

Similarly, can define bounce invariant:

$$J_{\parallel} = \oint d\ell \left[2m\left(E - \mu B(\ell)\right)\right]^{1/2}$$ longitudinal action

i.e. $V_{\parallel}^2(\ell) = V_{\parallel}^2(0) + V_{\perp}^2(0) - \mu B(\ell)$

etc.

squeeze → energy gain

N.B.: Treatment of adiabatic invariants given here corresponds to lowest order p.t. in $\frac{1}{\lambda}\frac{d\lambda}{dt}\Big/\omega < 1$

$\underbrace{}_{\text{"}O(\epsilon)\text{"}}$ here.

Note: Can also define 'mirror force'

$v_r \quad v_\theta \quad v_z$

$B_r \quad B_\theta \quad B_z$

$$F = \frac{q}{c} \underline{v} \times B$$

$$F_z = \frac{q}{c}\left(\cancel{v_r B_\theta} - v_\theta B_r\right)$$

$$\cong \frac{q}{c} \frac{v_\theta}{2} \frac{r\,\partial B_z}{\partial z}$$

$v_\theta \to v_\perp$

$r \to \rho$

$$F_z \cong \frac{q}{c}\frac{v_\theta r}{2}\frac{\partial B_z}{\partial z}$$

$$= \pm \frac{m v_\perp^2}{2B}\frac{\partial B}{\partial z} = \mp \mu \frac{\partial B}{\partial z}$$

depends on location in trajectory